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A new potential with the spectrum of an isotonic oscillator

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Abstract. With the method of factorisation, a new potential with free parameter is generated. This potential possesses exactly the same energy spectrum as the isotonic oscillator (harmonic oscillator with centripetal barrier). The wavefunctions and shift operators of the new system are also presented.

1. Introduction

It is well known that the class of analytically soluble systems in quantum mechanics is limited. Besides its practical value, a soluble model is often used as a test field for various purposes. Therefore finding more potentials, the energy spectra of which can be fully determined, is always of great interest. Recently, Abraham and Moses [1] developed an algorithm to construct a new soluble system in terms of the inverse scattering method. With the help of the Gelfand-Levitan equation one can generate a new soluble potential from a known one by adding or subtracting a finite number of eigenvalues and/or by changing the normalisations of a finite number of eigenfunctions. This approach is quite general. Then Mielnik [2] found that the traditional algebraic factorisation method of solving quantum mechanical problems [3] could also be used to construct a new potential with an energy spectrum coinciding with that of the harmonic oscillator. The factorisation method in this technique is not as general as the inverse scattering approach.

It is worthwhile to explore whether the factorisation method can be used for other systems. This paper will deal with a modified oscillator—an isotonic oscillator, i.e. a harmonic oscillator with a centripetal barrier. This system has been previously solved [4]. For our purposes, we give in § 2 an independent description to show how to solve this problem using the factorisation method. A new formula for the shift operators is found so that one can obtain the ground-state wavefunction by solving a first-order differential equation. In § 3 we present a new factorisation which allows us to construct a new potential with a free parameter. This new potential possesses the same energy spectrum as the isotonic oscillator. The wavefunctions and the shift operators of the new system are given in § 4. Finally we conclude this paper in § 5 with a short remark.

2. The isotonic oscillator

The isotonic oscillator is a system described by the following standardised potential:

$$V(x) = \tfrac{1}{2}x^2 + \tfrac{1}{2}gx^{-2} \tag{1}$$

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and the Hamiltonian operator is

$$H = -\tfrac{1}{2}\,\mathrm{d}^2/\mathrm{d}x^2 + \tfrac{1}{2}x^2 + \tfrac{1}{2}gx^{-2}. \tag{2}$$

To avoid 'fall to the centre' [5], the constant g should be greater than $-\frac{1}{4}$. For convenience (and recalling the radial Schrödinger equation for the spherically symmetric potential) we suppose

$$g = l(l+1) \tag{3}$$

where l can be any real number, but we will always take it to be non-negative.

The Hamiltonian operator (2) can be factorised as follows:

$$\begin{aligned} H(l) &= -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tfrac{1}{2}x^2 + \frac{l(l+1)}{2}\frac{1}{x^2} \\ &= b_l b_l^+ + l - \tfrac{1}{2} \\ &= b_{l+1}^+ b_{l+1} + l + \tfrac{3}{2} \end{aligned} \tag{4}$$

where the operators b_l and b_l^+ are defined as

$$b_l = \frac{1}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{l}{x} \qquad b_l^+ = \frac{1}{\sqrt{2}}\, -\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{l}{x}. \tag{5}$$

They satisfy the algebraic relation

$$b_l + b_{-l}^+ = \sqrt{2}x \qquad b_{-l}^+ - b_l^+ = \sqrt{2}l/x \tag{6}$$

and the commutative relation

$$\begin{aligned} &[b_l, b_l^+] = 1 + l/x^2 \\ &[b_l, b_{-l}] = -l/x^2 = [b_{-l}^+, b_l^+] \\ &[b_{-l}, b_l^+] = 1 = [b_l, b_{-l}^+]. \end{aligned} \tag{7}$$

Obviously the operators b_l and b_l^+ cannot be interpreted as annihilation and creation operators, but a short calculation reveals the following relations:

$$\begin{aligned} &H(l)b_l = b_l(H(l-1)-1) \\ &H(l-1)b_l^+ = b_l^+(H(l)+1) \end{aligned} \tag{8}$$

which implies that b_l and b_l^+ can be viewed as l-shift operators, or shift operators among two systems, $H(l)$ and $H(l-1)$. Using these operators one can obtain the solution of $H(l)$ from that of $H(l-1)$, and vice versa.

For example, suppose $\psi_n(l-1)$ is an eigenfunction of $H(l-1)$ with eigenvalue E_n

$$H(l-1)\psi_n(l-1) = E_n\psi_n(l-1) \tag{9}$$

then, according to equation (8), we have

$$\begin{aligned} H(l)b_l\psi_n(l-1) &= b_l(H(l-1)-1)\psi_n(l-1) \\ &= (E_n-1)b_l\psi_n(l-1) \end{aligned} \tag{10}$$

which means that $b_l\psi_n(l-1)$ is an eigenfunction of $H(l)$ with eigenvalue E_n-1. b_l is the operator lowering the energy level of $H(l-1)$ to $H(l)$.

If we change b_l, b_l^+ to b_{-l} and b_{-l}^+, we have another set of equations:

$$\begin{aligned} &H(l-1)b_{-l} = b_{-l}(H(l)-1) \\ &H(l)b_{-l}^+ = b_{-l}^+(H(l-1)+1). \end{aligned} \tag{11}$$

b_{-l}, b_{-l}^+ have a similar function as b_l and b_l^+, but the direction is different. Now the b_{-l} is lowering energy level of the $H(l)$ to produce that of $H(l-1)$.

Combining the two steps above, we obtain the correct shift operators for a single system, say, $H(l)$:

$$A_l^+ = b_{l+1}^+ b_{-(l+1)}^+ = b_{-l}^+ b_l^+ \qquad A_l = b_{-(l+1)} b_{l+1} = b_l b_{-l} \tag{12}$$

$$[H(l), A_l^+] = 2A_l^+ \qquad [H(l), A_l] = -2A_l. \tag{13}$$

These relations show that the energy spectrum of $H(l)$ is equally spaced. The spacing ΔE is two. Now we only need to know the ground state which should be annihilated by A_l:

$$A_l \psi_0(l) = 0 \tag{14}$$

or, to obtain a wavefunction satisfying the correct boundary condition,

$$b_{l+1}\psi_0(l) = \frac{1}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{l+1}{x}\psi_0(l) = 0. \tag{15}$$

This is a first-order differential equation which can be solved easily. We have the right ground-state wavefunction

$$\psi_0(l) = Nx^{l+1}\,\mathrm{e}^{-x^2/2} \tag{16}$$

where N is the normalisation number. The corresponding energy of the ground state can be calculated as follows:

$$\begin{aligned} H(l)\psi_0(l) &= (b_{l+1}^+ b_{l+1} + l + \tfrac{3}{2})\psi_0(l) \\ &= (l + \tfrac{3}{2})\psi_0(l) = E_0(l)\psi_0(l). \end{aligned} \tag{17}$$

Thus we know the lowest energy is

$$E_0(l) = l + \tfrac{3}{2}. \tag{18}$$

The general energy spectrum of the isotonic oscillator $H(l)$ is

$$E_n(l) = 2n + l + \tfrac{3}{2} \qquad n = 0, 1, 2, \ldots \tag{19}$$

and the corresponding wavefunctions can be constructed by using the raising operator A_l^+.

If we carry out the calculation, the raising operator turns out to be

$$A_l^+ = a^+ a^+ - l(l+1)/x^2 \tag{20}$$

where $a^+ = (1/\sqrt{2})(-\mathrm{d}/\mathrm{d}x + x)$ is the raising operator of the oscillator. This is just the form Camiz *et al* [6] used in their paper. Here we have factorised it so it is easier to obtain the ground state.

3. New factorisation

The isotonic oscillator $H(l)$ can be factorised in another way which will allow us to reach a new system. Let

$$H(l) = \frac{1}{2} - \frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{l(l+1)}{x^2} = d_l d_l^+ + l - \tfrac{1}{2} \tag{21}$$

and d_l, d_l^+ have the following form

$$d_l = b_l + \frac{1}{\sqrt{2}}\phi(x, l) = \frac{1}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{l}{x} + \phi$$
$$d_l^+ = b_l^+ + \frac{1}{\sqrt{2}}\phi(x, l) = \frac{1}{\sqrt{2}} - \frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{l}{x} + \phi \qquad (22)$$

where $\phi(x, l)$ is a real function.

Putting (22) back into (21), we obtain an equation for ϕ:

$$\mathrm{d}\phi/\mathrm{d}x + 2\beta(x)\phi + \phi^2 = 0$$
$$\beta(x) = x - l/x. \qquad (23)$$

This null Ricatti equation can be solved generally. The solution is

$$\phi(x, l) = x^{2l}\,\mathrm{e}^{-x^2}\,\gamma + {}_0^x\,\mathrm{d}t\,t^{2l}\,\mathrm{e}^{-t^2-1}. \qquad (24)$$

To avoid a singularity, we assume that all γ satisfy the constraint

$$|\gamma| > {}_0^\infty\,\mathrm{d}t\,t^{2l}\,\mathrm{e}^{-t^2} = \tfrac{1}{2}\Gamma(l+\tfrac{1}{2}). \qquad (25)$$

Finally, we have the operator d_l

$$d_l = \frac{1}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}x} + x - \frac{l}{x} + \phi(x, l) \qquad (26)$$

which provides a new factorisation. Obviously, when parameters $\gamma \to \infty$, all ϕ vanish and d_l returns to b_l. Since the commutator of d_l and d_l^+ no longer has the form of $1/x^2$, we now have the opportunity of generating a new potential with the same spectrum as the isotonic oscillator.

4. New potential

We construct a new Hamiltonian operator [2, 7]

$$\tilde{H}(l) = d_{l+1}^+ d_{l+1} + l + \tfrac{3}{2}$$
$$= H(l) - (\mathrm{d}/\mathrm{d}x)\phi(x, l+1) = -\mathrm{d}^2/\mathrm{d}x^2 + \tilde{V}(x, l) \qquad (27)$$

with a new potential

$$\tilde{V}(x, l) = V(x, l) - (\mathrm{d}/\mathrm{d}x)\phi(x, l+1) = \tfrac{1}{2}x^2 + \frac{l(l+1)}{2}\frac{1}{x^2} - \frac{\mathrm{d}}{\mathrm{d}x}\phi(x, l+1). \qquad (28)$$

We now prove that the new Hamiltonian $\tilde{H}(l)$ has the same spectrum as $H(l)$. Recalling (21), we see that

$$\tilde{H}(l)d_{l+1}^+ = (d_{l+1}^+ d_{l+1} + l + \tfrac{3}{2})d_{l+1}^+$$
$$= d_{l+1}^+(H(l+1) + 1) \qquad (29)$$

which means that we can obtain the eigenstates from that of $H(l+1)$ by using the raising operator d_{l+1}^{+}, and the corresponding energy levels are raised by a unit. Let $\psi_n(l+1)$ be the eigenfunction of $H(l+1)$

$$
\begin{aligned}
&H(l+1)\psi_n(l+1)=E_n(l+1)\psi_n(l+1)\\
&E_n(l+1)=2n+l+1+\tfrac{3}{2} \qquad n=0,1,2,\ldots
\end{aligned}
\tag{30}
$$

then

$$\tilde{\psi}_{n+1}(l)=Nd_{l+1}^{+}\psi_n(l+1) \tag{31}$$

is the eigenfunction of $\tilde{H}(l)$

$$\tilde{H}(l)\tilde{\psi}_{n+1}(l)=E_{n+1}(l)\tilde{\psi}_{n+1}(l) \tag{32}$$

with the eigenvalue

$$E_{n+1}(l)=2n+l+1+\tfrac{3}{2}+1=2(n+1)+l+\tfrac{3}{2} \qquad n=0,1,2,\ldots \tag{33}$$

which is the same as the energy level of $H(l)$ except for the ground state. The ground state of $\tilde{H}(l)$ can be easily obtained if we impose a condition

$$d_{l+1}\tilde{\psi}_0(l)=\frac{1}{\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}x}+x-\frac{l+1}{x}+\phi(x,l+1)\,\tilde{\psi}_0(l)=0 \tag{34}$$

which results in

$$\tilde{\psi}_0(l)=Nx^{l+1}\,\mathrm{e}^{-x^2/2}\exp-{}_0^x\,\phi(t,l+1)\,\mathrm{d}t \tag{35}$$

where again N denotes the normalisation constant. Now we have

$$\tilde{H}(l)\tilde{\psi}_0(l)=(d_{l+1}^{+}d_{l+1}+l+\tfrac{3}{2})\tilde{\psi}_0(l)=(l+\tfrac{3}{2})\tilde{\psi}_0(l) \tag{36}$$

so the energy of the ground state is

$$E_0(l)=l+\tfrac{3}{2}$$

which is the right value for $H(l)$. Thus the new system is completely solved.

Since $\psi_n(l+1)$ can be obtained from the eigenfunction of $H(l)$,

$$\psi_n(l+1)=Nb_{-(l+1)}^{+}\psi_n(l) \tag{37}$$

we can generate the eigenfunction of $\tilde{H}(l)$ directly through the eigenstate of $H(l)$:

$$\tilde{\psi}_{n+1}(l)=Nd_{l+1}^{+}b_{-(l+1)}^{+}\psi_n(l). \tag{38}$$

We can also have shift operators for the new system:

$$\tilde{A}_l^{+}=d_{l+1}^{+}b_{-(l+1)}^{+}b_{l+1}^{+}d_{l+1}=d_{l+1}^{+}A_{l+1}^{+}d_{l+1} \tag{39}$$

$$\tilde{\psi}_{n+1}(l)=N\tilde{A}_l^{+}\tilde{\psi}_n(l). \tag{40}$$

This shape of the raising operator is not difficult to understand. The operator sandwiched in between, $b_{-(l+1)}^{+}b_{l+1}^{+}$, is the raising operator of $H(l+1)$, and then d_{l+1}^{+} and d_{l+1} transfer it into the new system.

5. Conclusion

From the above we see that the factorisation method can be successfully applied to obtain a class of potentials which is exactly soluble and has the same point spectrum as the isotonic oscillator. Of course, another approach—the inverse scattering method—can be used here as well, and the resulting potential might be different. The Abraham-Moses algorithm implies that the set of potentials which support a given spectrum is uncountable, as in the oscillator case [8]. So different varieties of potential may supply more possibilities in some physical problems (as in the theory of quarkonium). It would be interesting to know whether there are some physical quantities apart from energy which can be used to characterise different members in the same class. In other words, the role of the parameter in the potential is a puzzle yet to be resolved.

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